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Ehrenfest times for classically chaotic systems

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We describe the quantum-mechanical spreading of a Gaussian wave packet by means of the semiclassical WKB approximation of Berry and Balazs [J. Phys. A **2**, 625 (1979)]. We find that the time scale τ on which this approximation breaks down in a chaotic system is larger than the Ehrenfest times considered previously. In one dimension $\tau=\frac{7}{6}\lambda^{-1}\ln(A/\hbar)$, with λ the Lyapunov exponent and A a typical classical action.

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According to Ehrenfest's theorem [1], the propagation of a quantum-mechanical wave packet is described for short times by classical equations of motion. The time scale at which this correspondence between quantum and classical dynamics breaks down is called the Ehrenfest time. If the classical dynamics is chaotic with Lyapunov exponent λ, then the Ehrenfest time τ is of order $\lambda^{-1}\ln(A/\hbar)$ (with A a typical classical action of the dynamical system) [2]. There is actually more than a single Ehrenfest time, corresponding to different types of semiclassical approximations. Although they differ only by a numerical coefficient, $\tau_i=c_i\lambda^{-1}\ln(A/\hbar)$, the structure of the wave function changes qualitatively from one time scale to the next.

Up to a time τ_1, with $c_1=1/6$, the initial coherent state will retain its Gaussian form with vanishing error in the limit $\hbar\rightarrow 0$ [3,4]. For longer times up to τ_2, with $c_2=1/2$, the uncertainty in the position and momentum of the particle remains small but the phase-space structure of the wave packet deviates strongly from a Gaussian. For times greater than τ_2 the wave function no longer has the form of a wave packet (this is the "mixing regime" of Refs. [5,6]), but up to a time τ_3 it can still be described semiclassically by the time-dependent WKB approximation of Berry and Balazs [7]. As we will show in this paper, the WKB representation implies $c_3=7/6$ for a single degree of freedom (with simple generalizations for higher dimensions). This is larger than the value $c_3=2/3$ obtained by Bouzouina and Robert [6] from a different semiclassical approximation.

Let us start with the Gaussian one-dimensional wave packet

$$\Psi(x)=\left(\frac{\alpha}{\pi\hbar}\right)^{1/4}\exp\left(i\frac{p_0x}{\hbar}+(i\beta-\alpha)\frac{(x-x_0)^2}{2\hbar}\right). \quad (1)$$

Initially $\beta(t=0)=0$ and $\alpha(t=0)=p_F/L$, where p_F and L are the typical classical momentum and length. The typical classical action is $A=p_FL$. The parameters $x_0(t),p_0(t)$ follow the classical trajectory for $\hbar\ll A$. We will measure the momentum and coordinate in units of p_F and L, respectively, so that $\alpha(0)=1$ and $A=1$. For chaotic dynamics with Lyapunov exponent λ one has $\alpha(t)\propto\exp(-2\lambda t)$, hence $\alpha\ll 1$ for $t\gg 1/\lambda$.

To describe the time evolution in phase space we consider the Wigner function

$$W(x,p)=\int\Psi\left(x+\frac{y}{2}\right)\Psi^*\left(x-\frac{y}{2}\right)e^{-ipy/\hbar}\frac{dy}{2\pi\hbar}$$
$$=\frac{1}{\pi\hbar}\exp\left(-\frac{\alpha(x-x_0)^2}{\hbar}-\frac{[p-p_0-\beta(x-x_0)]^2}{\alpha\hbar}\right). \quad (2)$$

The wave packet is centered at $x_0(t),p_0(t)$ and for $\alpha(t)\ll 1$ becomes highly elongated and tilted with slope $\Delta p/\Delta x\approx\beta$. It has length $l_\parallel=\sqrt{\hbar(1+\beta^2)/\alpha}$ and width $l_\perp=\sqrt{\hbar\alpha/(1+\beta^2)}$, so that the area in phase space is conserved exactly, $l_\parallel l_\perp=\hbar$. The Gaussian quantum wave packet satisfies the classical Liouville theorem.

The Gaussian form (1) takes into account the elongation of the wave packet, but not the curvature that develops in time and results in a bending of the packet. To describe the curvature we add an imaginary cubic term in the exponent in Eq. (1),

$$\Psi(x)=\left(\frac{\alpha}{\pi\hbar}\right)^{1/4}\exp\left(i\frac{p_0x}{\hbar}+\frac{(i\beta-\alpha)x^2}{2\hbar}+i\frac{\gamma x^3}{3\hbar}\right). \quad (3)$$

(For simplicity we have put $x_0=0$.) The cubic term leads to an appreciable phase shift over a length $l_\parallel\simeq(\hbar/\alpha)^{1/2}$ when $(\gamma/\hbar)(\hbar/\alpha)^{3/2}\gtrsim 1$, hence when $\alpha(t)\lesssim\hbar^{1/3}\gamma^{2/3}$.

For $\alpha\ll\hbar^{1/3}\gamma^{2/3}$ the Wigner function takes again a simple form, in terms of the Airy function Ai

$$W(x,p)=\frac{\alpha^{1/2}\exp(-\alpha x^2/\hbar)}{\pi\hbar^{1/2}(\gamma\hbar^2/4)^{1/3}}\mathrm{Ai}\left(\frac{p_0+\beta x+\gamma x^2-p}{(\gamma\hbar^2/4)^{1/3}}\right). \quad (4)$$

One can check that $W(x,p)\rightarrow\delta(x)\delta(p-p_0)$ when $\hbar\rightarrow 0$ (at fixed α), by means of the identity $\lim_{\varepsilon\rightarrow 0}\mathrm{Ai}(z/\varepsilon)/\varepsilon=\sqrt{\pi}\delta(z)$. At finite $\hbar$ the wave packet is extended along the curved line $p=p_0+\beta x+\gamma x^2$. Since p,p_0,x are of order unity, the two parameters β and γ are of order unity as well (in contrast to α, which is $\ll 1$). The transverse width is of order

$$l_\perp\approx\gamma^{1/3}\hbar^{2/3}(1+\beta^2)^{-1/2}. \quad (5)$$

The length of the packet remains at $l_\parallel\approx\sqrt{\hbar(1+\beta^2)/\alpha}$. Since now $l_\parallel l_\perp\gg\hbar$, the Liouville theorem no longer holds.

To obtain the Ehrenfest time, we parametrize time as

$$t=\frac{c}{\lambda}\ln\frac{1}{\hbar}. \tag{6}$$

The classical limit for a chaotic system means $\hbar\to 0$, $t\to\infty$ at fixed c. Different coefficients c follow from different semiclassical approximations. If we use the Gaussian wave packet (1), without the cubic term to account for the curvature, then we need $\alpha(t)\gg\hbar^{1/3}\gamma^{2/3}$. Since $\alpha\propto e^{-2\lambda t}\propto\hbar^{2c}$ we need $c<1/6$. The upper limit of c gives the first Ehrenfest time $\tau_1=\frac{1}{6}\lambda^{-1}\ln(1/\hbar)$.

The classical limit can be reached for longer times if we use the wave packet (3), including the cubic term. The dimensions of the packet for $t>\tau_1$ scale with $\hbar$ as

$$l_\perp\propto\hbar^{2/3}, \quad l_\parallel\propto\hbar^{1/2-c}. \tag{7}$$

For $c<1/2$ the length of the packet approaches zero in the classical limit. This upper limit of c gives the second Ehrenfest time $\tau_2=\frac{1}{2}\lambda^{-1}\ln(1/\hbar)$.

For $t>\tau_2$ the length of the wave packet exceeds the size of the system and is no longer small compared to the radius of curvature. For these large times we may adopt the semiclassical WKB approximation of Berry and Balazs [7]. Consider a curve in phase space $p(x)$ and a phase-space distribution $\rho(p(x),x)$. Both p and ρ evolve in accordance with classical equations of motion. For $t>\tau_2$ the function $p(x)$ is multivalued with an exponentially large number of branches $\sim\exp[\lambda(t-\tau_2)]$. The quantum wave function in this "mixing" regime has the form

$$\Psi(x)=\sum_k f_k(x)\exp[i\sigma_k(x)/\hbar]. \tag{8}$$

The summation over k accounts for the different branches of the multivalued function $p(x)$. The two functions f and σ are related for $\hbar\to 0$ to p and ρ by the correspondence principle

$$\frac{d\sigma}{dx}=p(x), \quad f=\sqrt{\rho(p,x)}. \tag{9}$$

An explicit description of the evolution of the wave function (8) for quantum maps can be found in Ref. [8].

Near the point x_b at which $p(x)$ bifurcates into two branches, one has $p=p_b\pm a\sqrt{x-x_b}$, $\rho=b/\sqrt{x-x_b}$. The wave function there is

$$\Psi=(\hbar/a)^{1/3}b^{1/2}\,\mathrm{Ai}[(a/\hbar)^{2/3}(x-x_b)]e^{ip_bx}, \tag{10}$$

up to an overall phase. The phase difference between the bifurcation points can be determined from Eqs. (8) and (9). Because the curve $p(x)$ is not closed, there is no analog of the Bohr-Sommerfeld quantization rule.

The Wigner function corresponding to the wave function (8), being quadratic in Ψ, contains both diagonal ($W_{kk}\propto|f_k|^2$) and oscillating nondiagonal ($W_{km}\propto f_k^\dagger f_m$) contributions. Far from bifurcations, the diagonal contributions to the Wigner function read

$$\begin{aligned}W_{kk}(x,p)&=\int\exp\left(\frac{iy(\sigma'-p)}{\hbar}+\frac{iy^3\sigma'''}{24\hbar}\right)\frac{|f(x)|^2dy}{2\pi\hbar}\\&=\frac{2}{\sqrt{\pi}}\left(\frac{1}{\hbar^2\sigma'''}\right)^{1/3}|f(x)|^2\,\mathrm{Ai}\left(\frac{2(\sigma'-p)}{(\hbar^2\sigma''')^{1/3}}\right).\end{aligned} \tag{11}$$

We have made a Taylor expansion of $\sigma(x\pm y/2)$ and neglected the difference between $f(x\pm y/2)$ and $f(x)$.

If we parametrize time as in Eq. (6) we have for both $l_\parallel$ and $l_\perp$ the same scaling with $\hbar$ as in Eq. (7). The range of validity of Eq. (8) is limited by the condition that the different branches should be distinguishable. This requires that the different parts of the curve $p(x)$ in phase space should not get closer than $l_\perp$. Their spacing is of order $1/l_\parallel$ (assuming a uniform filling of phase space), hence

$$l_\parallel l_\perp\ll 1\Rightarrow\hbar^{7/6-c}\ll 1. \tag{12}$$

The upper limit of 7/6 for c leads to the third Ehrenfest time

$$\tau_3=\frac{7}{6\lambda}\ln\frac{1}{\hbar}. \tag{13}$$

The third derivative σ''' in Eq. (11) vanishes at the points of inflection of the curve $p(x)$. In order to find the Wigner function there, one should expand $\sigma(x\pm y/2)$ up to terms of order y^5. This leads to a different scaling $l_\perp\propto\hbar^{4/5}$ of the width of the Wigner function near the inflection points. Because these are isolated points, they will not contribute to the matrix elements of nonsingular operators (containing only smooth functions of x and p). This different scaling should therefore not affect the Ehrenfest time (13).

The nondiagonal contributions W_{km} to the Wigner function lead to the "ghost curves" discussed in Ref. [9]. (Ghost curves are regions of large values of the Wigner function which do not correspond to classical trajectories.) The Wigner function near these curves is given by the same Airy function as in Eq. (11), but in addition acquires a strongly oscillating factor. Due to these oscillations the nondiagonal terms do not contribute to the matrix elements of nonsingular operators. (They may play a role in the decoherence by the environment [10].) At $t\gtrsim\tau_3$ the ghost curves merge with the (multivalued) curve $p(x)$ and become indistinguishable.

The time scale (13) for the breakdown of the WKB approximation is greater than the Ehrenfest time $\frac{2}{3}\lambda^{-1}\ln(1/\hbar)$ in the mixed regime obtained in Ref. [6]. That shorter time scale may signal the breakdown of the series expansion $\sigma_k(x)\to\Sigma_{j=0}\sigma_{kj}(x)\hbar^j$. Then Eq. (9) would no longer hold, but for $t<\tau_3$ the representation (8) with a renormalized function $\sigma_k(x)$ would still be valid.

So far we have discussed a one-dimensional (1D) chaotic system, which in general can be represented by an area preserving map [8]. A familiar example is the kicked rotator [11]. For mesoscopic quantum dots, however, a more relevant model is the d-dimensional ($d=2,3$) Schrödinger equation with a smooth potential $V(\vec{r})$. The Gaussian wave packet then takes the form

$$\Psi(\vec{r}) \propto \exp\left[\frac{i}{\hbar}\left(S(\vec{r}_0(t)) + \vec{p}_0 \cdot \vec{x} + \frac{\zeta_{ln}}{2} x_l x_n\right)\right]. \qquad (14)$$

Here S is the action for the classical trajectory $\vec{r}_0(t)$ and we have defined $\vec{p}_0 = m\dot{\vec{r}}_0$, $\vec{x} = \vec{r} - \vec{r}_0$, $\zeta_{ln} = \beta_{ln} + i\alpha_{ln}$. As before, we rescale the momentum and coordinate such that the typical classical action $A = 1$. Initially, $\zeta_{ln} \simeq i\delta_{ln}$. Similar to the one-dimensional case, α_{ln} defines the form of the packet in coordinate space and $\beta_{ln} = \Delta p_l / \Delta x_n$ give the angles in phase space. Substituting the wave function (14) into the Schrödinger equation one finds Newton's equation of motion for $\vec{r}_0$. The spreading of the wave packet in phase space is described by

$$-\dot{\zeta}_{ln} = \frac{1}{m}\zeta_{lk}\zeta_{kn} + \left.\frac{\partial^2 V}{\partial r_l \partial r_n}\right|_{\vec{r}=\vec{r}_0}. \qquad (15)$$

This is the equation describing the spreading in phase space of a small Gaussian bunch of classical particles.

The Wigner function corresponding to the wave function (14) has the Gaussian form $W \propto \exp(-Q_l M_{ln} Q_n/\hbar)$, where $\vec{Q} = (\vec{r} - \vec{r}_0, \vec{p} - \vec{p}_0)$ is a vector in $2d$-dimensional phase space. The d Lyapunov exponents λ_i ($i = 1,2,\ldots,d$) govern the large-time behavior of the eigenvalues $m_i = 1/m_{2d-i+1} \propto \exp(2\lambda_i t)$ of the real symmetric matrix M. Because of energy conservation one Lyapunov exponent vanishes. We order the λ's from large to small, so that λ_1 is the largest and $\lambda_d = 0$.

The wave packet remains Gaussian (preserving the volume $\propto \hbar^d$ in phase space) until the curvature starts to play a role (via a cubic term in the action). The corresponding Ehrenfest time $\tau_1 = \frac{1}{6}\lambda_1^{-1}\ln(1/\hbar)$ is the same as in 1D, only now it is defined through the largest Lyapunov exponent λ_1. The second Ehrenfest time, when the length of the packet exceeds the size of the system, also has the same form $\tau_2 = \frac{1}{2}\lambda_1^{-1}\ln(1/\hbar)$.

The third time τ_3 is different for $d = 2,3$ from the 1D case. Instead of Eq. (7), one now has

$$l_\perp^{(i)} \propto \hbar^{2/3}, \quad l_\parallel^{(i)} \propto \hbar^{1/2} e^{\lambda_i t}, \quad i = 1,2,\ldots,d-1. \qquad (16)$$

The longitudinal dimensions $l_\parallel^{(i)}$ correspond to eigenvalues m_i with $1 \leqslant i \leqslant d-1$, and the transverse dimensions $l_\perp^{(i)}$ to m_i with $d+2 \leqslant i \leqslant 2d$. The two unit eigenvalues $m_d = m_{d+1} = 1$ contribute another factor $\sqrt{\hbar}$ each to the total volume $\mathcal{V}$ in phase space covered by the wave packet

$$\mathcal{V} = \hbar \prod_{i=1}^{d-1} l_\perp^{(i)} l_\parallel^{(i)} \propto \hbar^{7d/6-1/6} e^{\lambda_{\rm tot} t}, \quad \lambda_{\rm tot} = \sum_{i=1}^{d-1} \lambda_i. \qquad (17)$$

The available area $\mathcal{V}_{\max}$ is restricted to a shell of constant energy with thickness $\sqrt{\hbar}$, hence $\mathcal{V}_{\max} \propto \sqrt{\hbar}$. We require $\mathcal{V} \lesssim \mathcal{V}_{\max}$ for the semiclassical approximation, which leads to the Ehrenfest time

$$\tau_3 = \frac{7d-4}{6\lambda_{\rm tot}} \ln\frac{A}{\hbar}, d \geqslant 2. \qquad (18)$$

In conclusion, we examined different time scales $\tau_i = c_i \lambda^{-1} \ln(1/\hbar)$ for the breakdown of different types of semiclassical approximations. These Ehrenfest times differ only by a numerical coefficient c_i, which may seem insignificant. However, this difference is actually a signal of a different power law scaling with $\hbar$ of the volume $\mathcal{V}$ in phase space covered by the wave packet. For short times Liouville's theorem dictates $\mathcal{V} \propto \hbar$. For long times [parameterized as $t = (c/\lambda)\ln(1/\hbar)$] the WKB approximation gives $\mathcal{V} \propto \hbar^{7/6-c}$ for a one-dimensional quantum map (such as the kicked rotator) and $\mathcal{V} \propto \hbar^{7d/6-1/6-c}$ for a d-dimensional conservative system. These different power laws reflect the fundamental change in the structure of the wave function with increasing time and should, therefore, have observable consequences. Two possible applications are the Loschmidt echo [12] and the quantum shot noise [13], where the Ehrenfest time plays a key role.

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